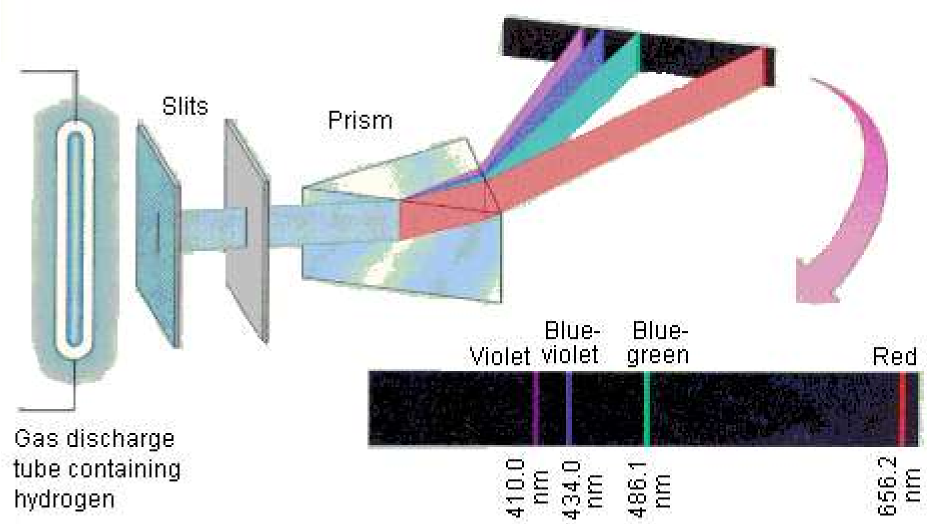
Emission Spectrum of Hydrogen

When an electric current is passed through a glass tube that contains hydrogen gas at low pressure the tube gives off blue light. When this light is passed through a prism (as shown in the figure below), four narrow bands of bright light are observed against a black background.



These narrow bands have the characteristic wavelengths and colors shown in the table below.

Wavelength	**Color**
656.2	red
486.1	blue-green
434.0	blue-violet
410.1	violet

Four more series of lines were discovered in the emission spectrum of hydrogen by searching the infrared spectrum at longer wavelengths and the ultraviolet spectrum at shorter wavelengths. Each of these lines fits the same general equation, where n_1 and n_2 are integers and R_H is 1.09678 x 10^{-2} nm^{-1}.

$$\frac{1}{\lambda} = R_H\left[\frac{1}{{n_1}^2} - \frac{1}{{n_2}^2}\right]$$

Explanation of the Emission Spectrum

Max Planck presented a theoretical explanation of the spectrum of radiation emitted by an object that glows when heated. He argued that the walls of a glowing solid could be imagined to contain a series of resonators that oscillated at different frequencies. These resonators gain energy in the form of heat from the walls of the object and lose energy in the form of electromagnetic radiation. The energy of these resonators at any moment is proportional to the frequency with which they oscillate.

To fit the observed spectrum, Planck had to assume that the energy of these oscillators could take on only a limited number of values. In other words, the spectrum of energies for these oscillators was no longer continuous. Because the number of values of the energy of these oscillators is limited, they are theoretically "countable." The energy of the oscillators in this system is therefore said to be ***quantized***. Planck introduced the notion of quantization to explain how light was emitted.

Albert Einstein extended Planck's work to the light that had been emitted. At a time when everyone agreed that light was a wave (and therefore continuous), Einstein suggested that it behaved as if it was a stream of small bundles, or packets, of energy. In other words, light was also quantized. Einstein's model was based on two assumptions. First, he assumed that light was composed of **photons**, which are small, discrete bundles of energy. Second, he assumed that the energy of a photon is proportional to its frequency.

$E = h\nu$

h is a constant known as Planck's constant, which is equal to 6.626×10^{-34} J-s.

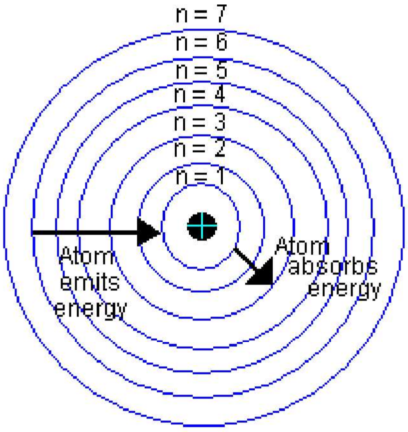
The fact that hydrogen atoms emit or absorb radiation at a limited number of frequencies implies that these atoms can only absorb radiation with certain energies. This suggests that there are only a limited number of energy levels within the hydrogen atom. These energy levels are countable. The energy levels of the hydrogen atom are quantized.

The Bohr Model of the Atom

Niels Bohr proposed a model for the hydrogen atom that explained the spectrum of the hydrogen atom. The **Bohr model** was based on the following assumptions.

- The electron in a hydrogen atom travels around the nucleus in a circular orbit.
- The energy of the electron in an orbit is proportional to its distance from the nucleus. The further the electron is from the nucleus, the more energy it has.
- Only a limited number of orbits with certain energies are allowed. In other words, the orbits are quantized.
- The only orbits that are allowed are those for which the angular momentum of the electron is an integral multiple of $h/2\pi$.
- Light is absorbed when an electron jumps to a higher energy orbit and emitted when an electron falls into a lower energy orbit.
- The energy of the light emitted or absorbed is exactly equal to the difference between the energies of the orbits.

Some of the key elements of this hypothesis are illustrated in the figure below.



Three points deserve particular attention. First, Bohr recognized that his first assumption violates the principles of classical mechanics. But he knew that it was impossible to explain the spectrum of the hydrogen atom within the limits of classical physics. He was therefore willing to assume that one or more of the principles from classical physics might not be valid on the atomic scale.

Second, he assumed there are only a limited number of orbits in which the electron can reside. He based this assumption on the fact that there are only a limited number of lines in the spectrum of the hydrogen atom and his belief that these lines were the result of light being emitted or absorbed as an electron moved from one orbit to another in the atom.

Finally, Bohr restricted the number of orbits on the hydrogen atom by limiting the allowed values of the angular momentum of the electron. Any object moving along a straight line has a *momentum* equal to the product of its mass (m) times the velocity (v) with which it moves. An object moving in a circular orbit has an *angular momentum* equal to its mass (m) times the velocity (v) times the radius of the orbit (r). Bohr assumed that the angular momentum of the electron can take on only certain values, equal to an integer times Planck's constant divided by 2π.

$mvr = n\left[\frac{h}{2\pi}\right]$ where n= 1, 2, 3, 4, ...

Bohr then used classical physics to show that the energy of an electron in any one of these orbits is inversely proportional to the square of the integer $\boldsymbol{n}$.

The difference between the energies of any two orbits is therefore given by the following equation.

$$\Delta E = R_H\left[\frac{1}{{n_1}^2} - \frac{1}{{n_2}^2}\right]$$

In this equation, n_1 and n_2 are both integers and R_H is the proportionality constant known as the Rydberg constant.

Planck's equation states that the energy of a photon is proportional to its frequency.

$E = hv$

Substituting the relationship between the frequency, wavelength, and the speed of light into this equation suggests that the energy of a photon is inversely proportional to its wavelength. The inverse of the wavelength of electromagnetic radiation is therefore directly proportional to the energy of this radiation.

By properly defining the units of the constant, R_H, Bohr was able to show that the wavelengths of the light given off or absorbed by a hydrogen atom should be given by the following equation.

$$\frac{1}{\lambda} = R_H \left[\frac{1}{{n_1}^2} - \frac{1}{{n_2}^2}\right]$$

Thus, once he introduced his basic assumptions, Bohr was able to derive an equation that matched the relationship obtained from the analysis of the spectrum of the hydrogen atom.

According to the Bohr model, the wavelength of the light emitted by a hydrogen atom when the electron falls from a high energy (n= 4) orbit into a lower energy (n= 2) orbit.

Substituting the appropriate values of R_H, n_1, and n_2 into the equation shown above gives the following result.

$$\frac{1}{\lambda} = (\ 1.09678x10^{-2}\ nm^{-1})[\frac{1}{2_1^2} - \frac{1}{4_2^2}]$$

Solving for the wavelength of this light gives a value of 486.3 nm, which agrees with the experimental value of 486.1 nm for the blue line in the visible spectrum of the hydrogen atom.

Wave-Particle Duality

The theory of **wave-particle duality** developed by Louis-Victor de Broglie eventually explained why the Bohr model was successful with atoms or ions that contained one electron. It also provided a basis for understanding why this model failed for more complex systems. Light acts as both a particle and a wave. In many ways light acts as a wave, with a characteristic frequency, wavelength, and amplitude. Light carries energy as if it contains discrete photons or packets of energy.

When an object behaves as a particle in motion, it has an energy proportional to its mass (m) and the speed with which it moves through space (s).

$E = ms^2$

When it behaves as a wave, however, it has an energy that is proportional to its frequency:

$$E = hv = \frac{hs}{\lambda}$$

By simultaneously assuming that an object can be both a particle and a wave, de Broglie set up the following equation.

$$ms^2 = \frac{hs}{\lambda}$$

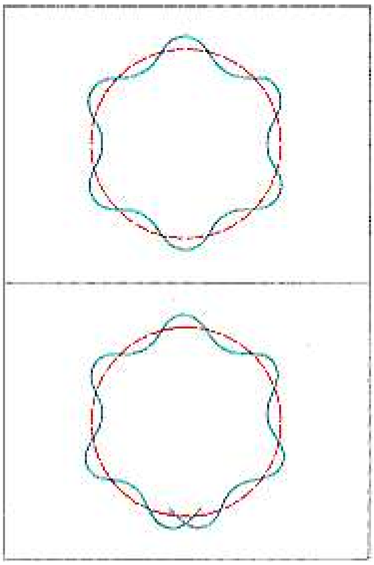
By rearranging this equation, he derived a relationship between one of the wave-like properties of matter and one of its properties as a particle.

As noted in the previous section, the product of the mass of an object multiplied the speed with which it moves is the *momentum* (p) of the particle. Thus, the **de Broglie equation** suggests that the wavelength (λ) of any object in motion is inversely proportional to its momentum.

$$\lambda = \frac{h}{p}$$

De Broglie concluded that most particles are too heavy to observe their wave properties. When the mass of an object is very small, however, the wave properties can be detected experimentally. De Broglie predicted that the mass of an electron was small enough to exhibit the properties of both particles and waves. In 1927 this prediction was confirmed when the diffraction of electrons was observed experimentally by C. J. Davisson.

De Broglie applied his theory of wave-particle duality to the Bohr model to explain why only certain orbits are allowed for the electron. He argued that only certain orbits allow the electron to satisfy both its particle and wave properties at the same time because only certain orbits have a circumference that is an integral multiple of the wavelength of the electron, as shown in the figure below.



The Bohr Model vs. Reality

At first glance, the Bohr model looks like a two-dimensional model of the atom because it restricts the motion of the electron to a circular orbit in a two-dimensional plane. In reality the Bohr model is a one-dimensional model, because a circle can be defined by specifying only one dimension: its radius, *r*. As a result, only one coordinate (*n*) is needed to describe the orbits in the Bohr model.

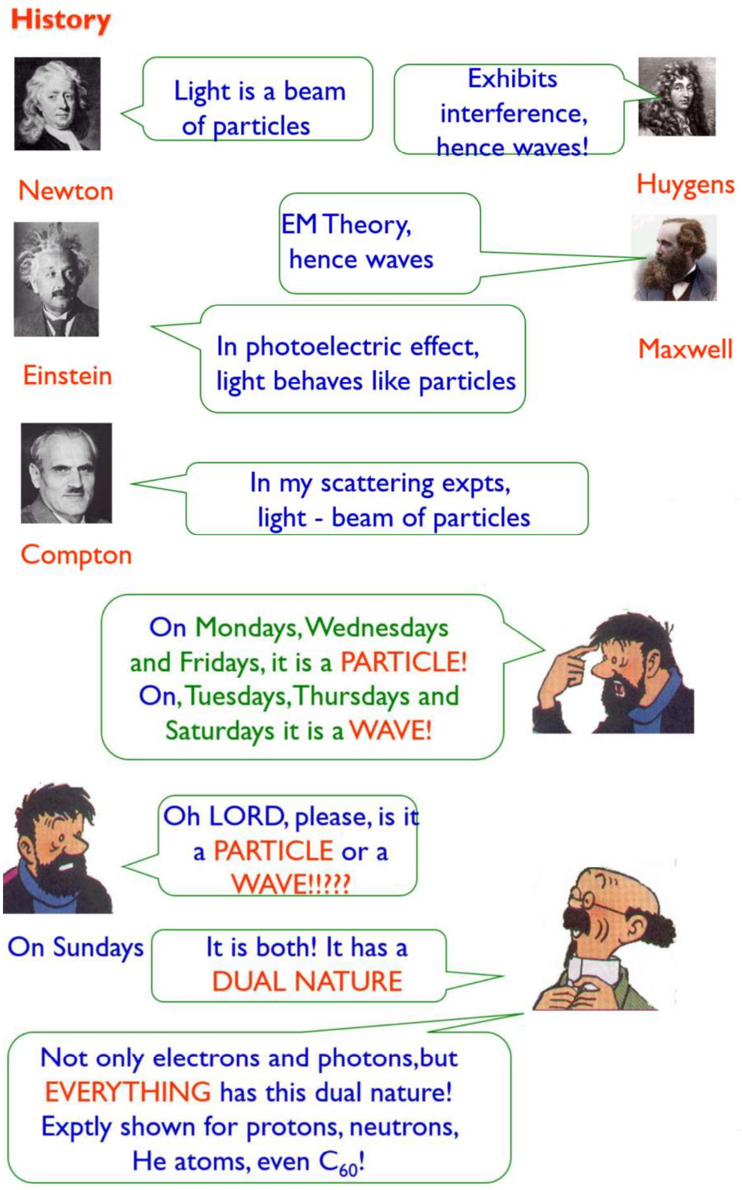
Unfortunately, electrons aren't particles that can be restricted to a one-dimensional circular orbit. They act to some extent as waves and therefore exist in three-dimensional space. The Bohr model works for one-electron atoms or ions only because certain factors present in more complex atoms are not present in these atoms or ions. To construct a model that describes the distribution of electrons in atoms that contain more than one electron we have to allow the electrons to occupy three-dimensional space. We therefore need a model that uses three coordinates to describe the distribution of electrons in these atoms.

Wave Functions and Orbitals

We still talk about the Bohr model of the atom even if the only thing this model can do is explain the spectrum of the hydrogen atom because it was the last model of the atom for which a simple physical picture can be constructed. It is easy to imagine an atom that consists of solid electrons revolving around the nucleus in circular orbits.

Erwin Schrödinger combined the equations for the behavior of waves with the de Broglie equation to generate a mathematical model for the distribution of electrons in an atom. The advantage of this model is that it consists of mathematical equations known as *wave functions* that satisfy the requirements placed on the behavior of electrons. The disadvantage is that it is difficult to imagine a physical model of electrons as waves.

The Schrödinger model assumes that the electron is a wave and tries to describe the regions in space, or **orbitals**, where electrons are most likely to be found. Instead of trying to tell us where the electron is at any time, the Schrödinger model describes the probability that an electron can be found in a given region of space at a given time. This model no longer tells us where the electron is; it only tells us where it might be.

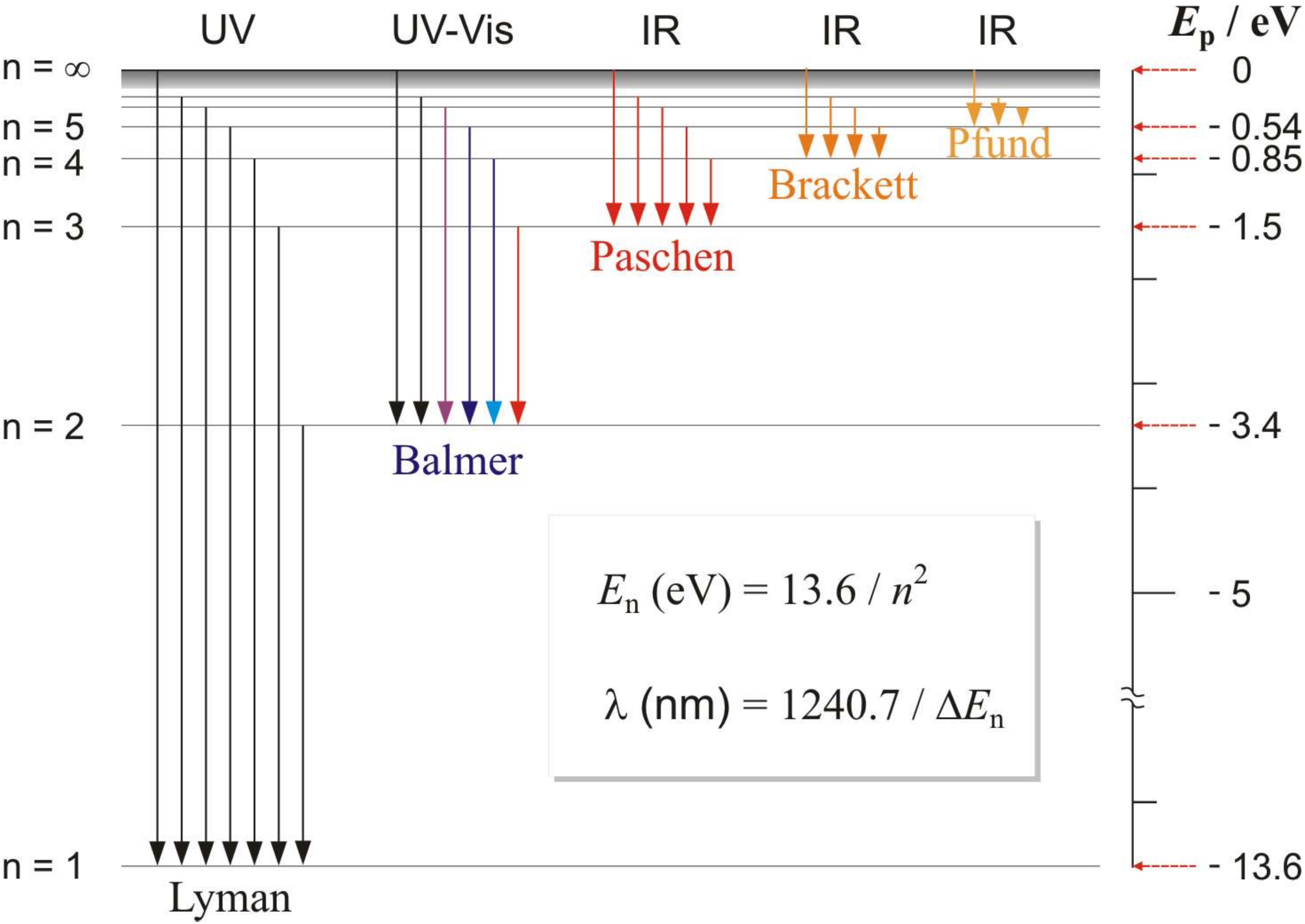


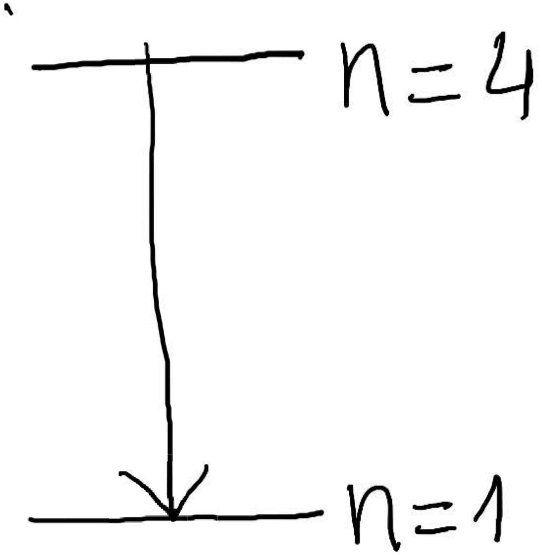
Reference credit: Prof. K. L. Sebastian, IISc Bangalore

Hydrogen emission spectral lines

- These slides are for the UG-students of my class and are not intended for circulation publicly.
- Efforts have been made to credit all the sources from where the material has been taken.

Johann Jakob Balmer 1885
Johannes Rydberg 1888
Niels Bohr 1913


UV
UV-Vis
IR
IR
IR
E_p / eV
n = ∞
n = 5
n = 4
n = 3
n = 2
n = 1
0
- 0.54
- 0.85
- 1.5
- 3.4
- 5
- 13.6
Pfund
Brackett
Paschen
Balmer
Lyman
E_n (eV) = 13.6 / n^2
λ (nm) = 1240.7 / ΔE_n



Lymann Series

$$\Delta E_n = -\frac{13.6}{4^2} + \frac{13.6}{1^2}$$

$$= 13.6 - \frac{13.6}{16}$$

$$= 13.6 - 0.85$$

$$= 12.75$$

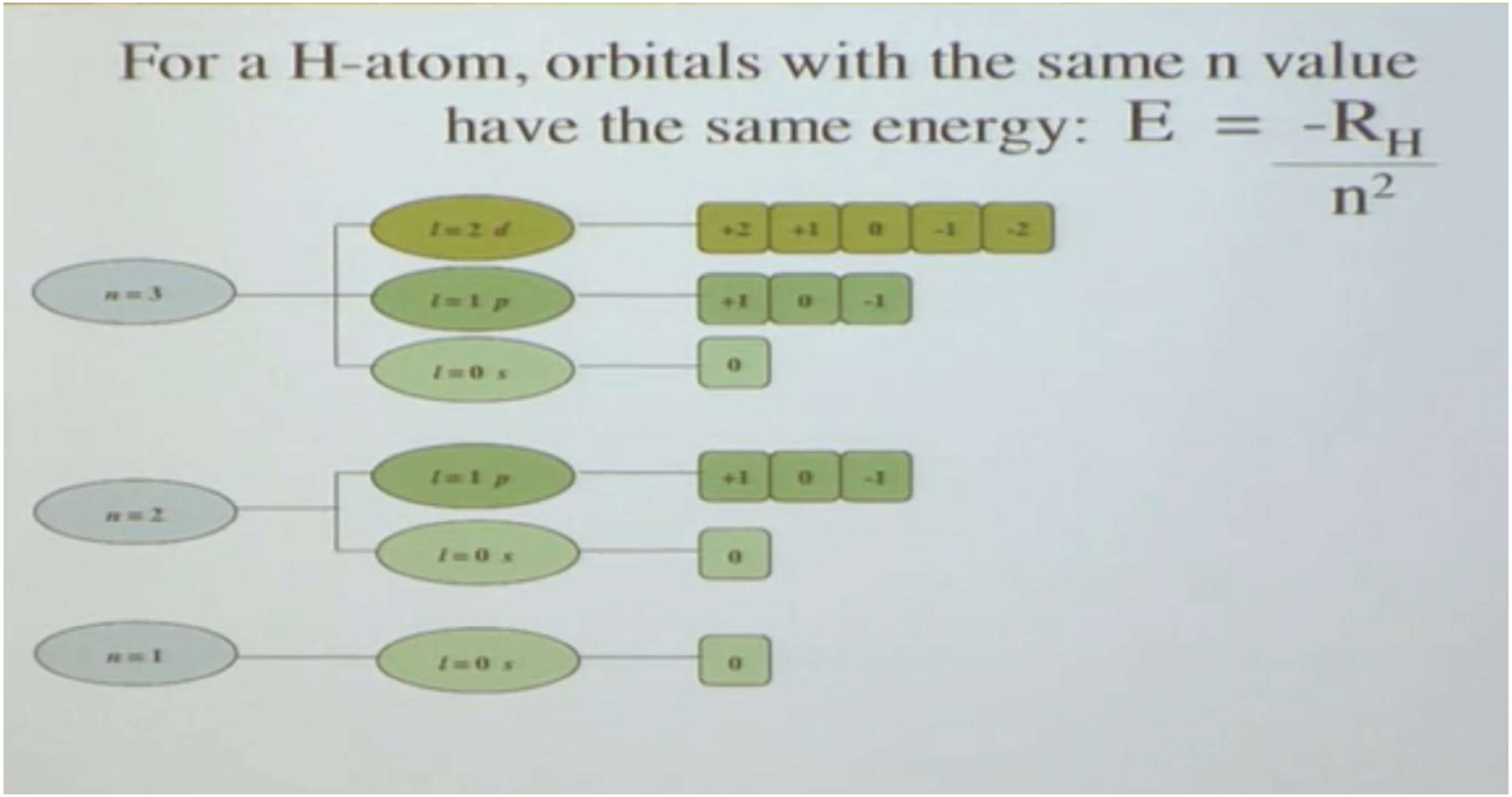
$$\lambda = \frac{1240.7}{12.75} \text{ nm}$$

$$= 97.3 \text{ nm}$$

Solving the Schrödinger Equation provides values for E_n and $\Psi(r,\theta,\phi)$.

A total of three quantum numbers are needed to describe a wavefunction in three-dimensions:

1. n = principal quantum number
2. l = angular momentum quantum number
3. m_l = magnetic quantum number



For a H-atom, orbitals with the same n value have the same energy: $E = \frac{-R_H}{n^2}$

R_H= Rydberg Constant

MIT OpenCourseware: Prof. Catherine Drennan
https://ocw.mit.edu/courses/5-111sc-principles-of-chemical-science-fall-2014/

How many different values of the quantum number are possible when n = 14 ?

How many different values of m_l are allowed for an electron in a 9d subshell?

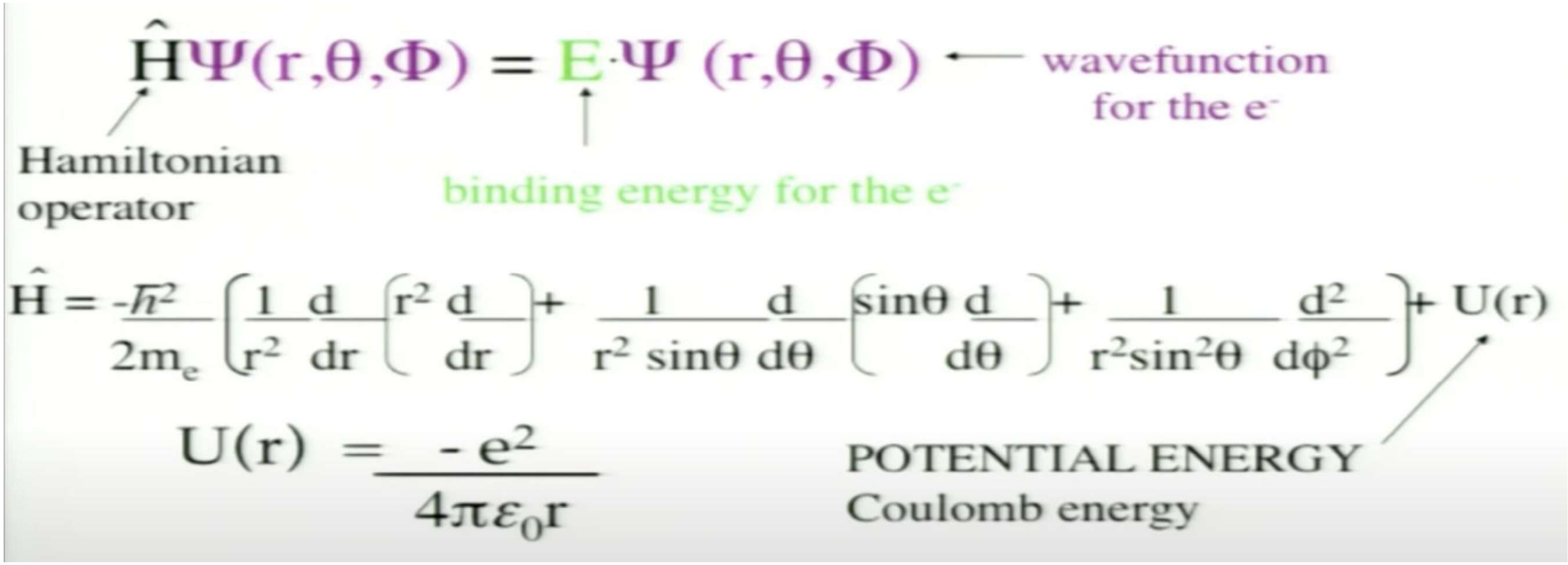
How many values of m_l are allowed for a 5s subshell?

Erwin Schrödinger (born August 12, 1887, Vienna, Austria—died January 4, 1961, Vienna) was an Austrian theoretical physicist who contributed to the wave theory of matter and to other fundamentals of quantum mechanics. He shared the 1933 Nobel Prize for Physics with British physicist P.A.M. Dirac.

Schrödinger introduced a theory describing the behaviour of sub-atomic systems by a wave equation that is now known as the Schrödinger equation. The solutions to Schrödinger's equation, unlike the solutions to Newton's equations, are wave functions that can only be related to the probable occurrence of physical events. The definite and readily visualized sequence of events of the planetary orbits of Newton is, in quantum mechanics, replaced by the more abstract notion of probability.

https://www.britannica.com/biography/Erwin-Schrodinger

Representing Atomic Orbitals with the Help of Wavefunctions



$$\hat{H}\Psi(r,\theta,\Phi) = E\cdot\Psi(r,\theta,\Phi)$$

← wavefunction for the e^-

Hamiltonian operator

binding energy for the e^-

$$\hat{H} = \frac{-\hbar^2}{2m_e}\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2}{d\phi^2}\right) + U(r)$$

$$U(r) = \frac{-e^2}{4\pi\varepsilon_0 r}$$

POTENTIAL ENERGY
Coulomb energy

For a hydrogen atom:

$$\hat{H}\Psi = E\cdot\Psi$$

$$\hat{H}\Psi = \underbrace{-\frac{1}{n^2}\,\frac{me^4}{8\varepsilon_0^2h^2}}\cdot\Psi$$

H atom orbital or wavefunction

E ≡ binding energy of the electron to the nucleus

$m =$ **m_e = electron mass**

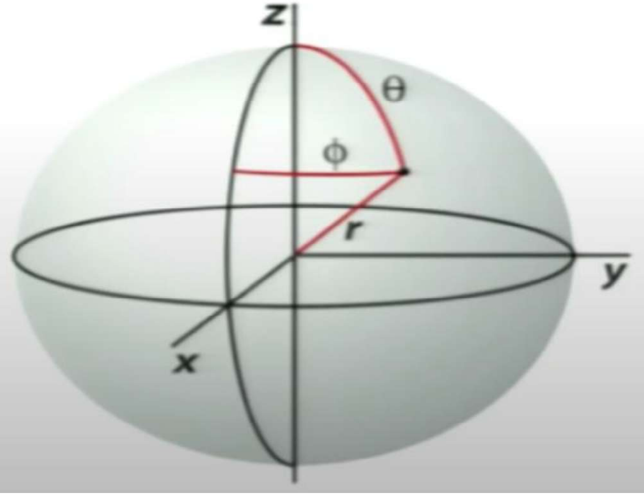
$e =$ **electron charge**

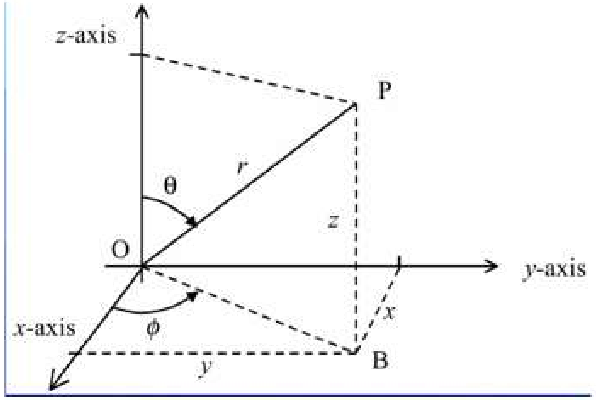
ε_0 = permittivity constant

h = Planck's constant

To consider the 3d shapes of orbital we can consider the total wavefunction **Ψ** as a product of a radial wavefunction **R** and an angular wavefunction **Y**

$$\Psi_{nlm}(r,\theta,\phi)] = \underbrace{R_{nl}(r)}_{\text{radial }\Psi} \times \underbrace{Y_{lm}(\theta,\phi)}_{\text{angular }\Psi}$$

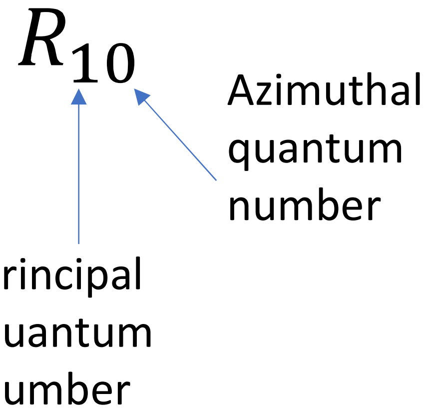




SIZE determined by $R_{nl}(r)$ = radial part of the wavefunction

SHAPE determined by $Y_{lm}(\theta,\phi)$ = angular part of the wavefunction(spherical harmonics)

Radial Part



R_{10}

Azimuthal quantum number

Principal quantum number

1s $R_{10} = 2\left(\frac{Z}{a_0}\right)^{\frac{3}{2}} e^{-Zr/a_0}$

2p $R_{21} = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{\frac{3}{2}}\left(\frac{Zr}{a_0}\right) e^{-Zr/2a_0}$

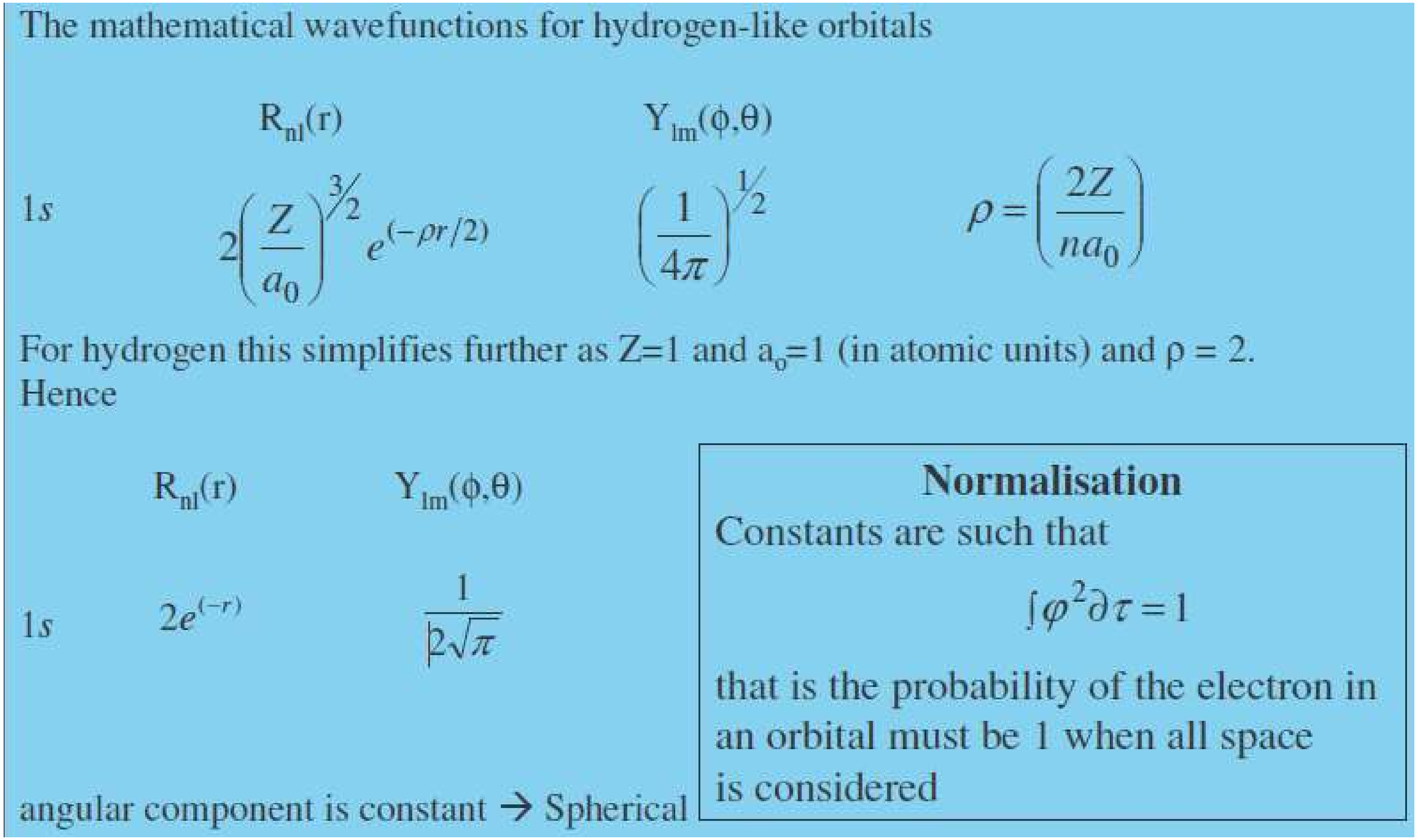
2s $R_{20} = 2\left(\frac{Z}{2a_0}\right)^{\frac{3}{2}}\left(1-\frac{Zr}{2a_0}\right) e^{-Zr/2a_0}$

3d $R_{32} = \frac{2\sqrt{2}}{27\sqrt{5}}\left(\frac{Z}{3a_0}\right)^{\frac{3}{2}}\left(\frac{Zr}{a_0}\right)^{2} e^{-Zr/3a_0}$

3p $R_{31} = \frac{4\sqrt{2}}{3}\left(\frac{Z}{3a_0}\right)^{\frac{3}{2}}\left(\frac{Zr}{a_0}\right)\left(1-\frac{Zr}{6a_0}\right) e^{-Zr/3a_0}$

3s $R_{30} = 2\left(\frac{Z}{3a_0}\right)^{\frac{3}{2}}\left(1-\frac{2Zr}{3a_0}+\frac{2\left(Zr\right)^{2}}{27a_0^{2}}\right) e^{-Zr/3a_0}$

1s orbital



The mathematical wavefunctions for hydrogen-like orbitals

	$R_{nl}(r)$	$Y_{lm}(\phi,\theta)$	
1s	$2\left(\frac{Z}{a_0}\right)^{3/2} e^{(-\rho r/2)}$	$\left(\frac{1}{4\pi}\right)^{1/2}$	$\rho = \left(\frac{2Z}{na_0}\right)$

For hydrogen this simplifies further as Z=1 and a_o=1 (in atomic units) and ρ = 2.
Hence

	$R_{nl}(r)$	$Y_{lm}(\phi,\theta)$
1s	$2e^{(-r)}$	$\frac{1}{2\sqrt{\pi}}$

Normalisation

Constants are such that

$$\int \varphi^2 \partial\tau = 1$$

that is the probability of the electron in an orbital must be 1 when all space is considered

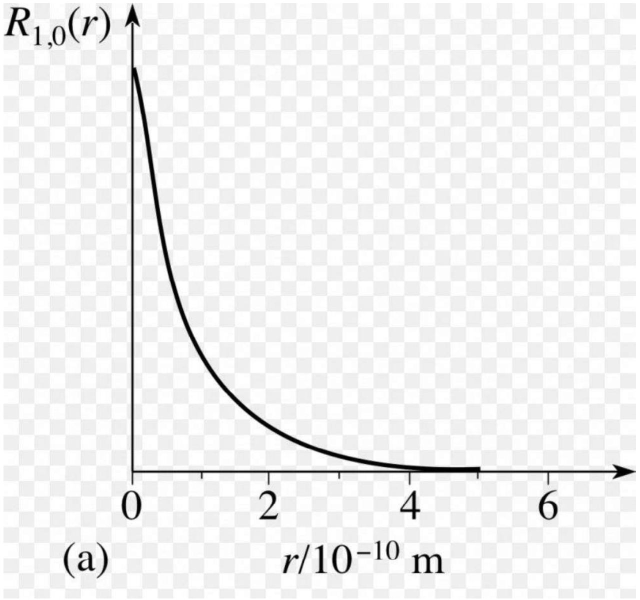
angular component is constant → Spherical

Graphical representation of a Radial Wavefunction

- R(r) of the 1*s* orbital of H
- R(r) = 2$e^{(-r)}$

- It decays exponentially with r
- It has a maximum at r = 0

- **R(r) has no physical meaning. R(r)^2 does. It represents the probability density of the electron at a point in space**
- **R(r)^2 is a maximum at r = 0**



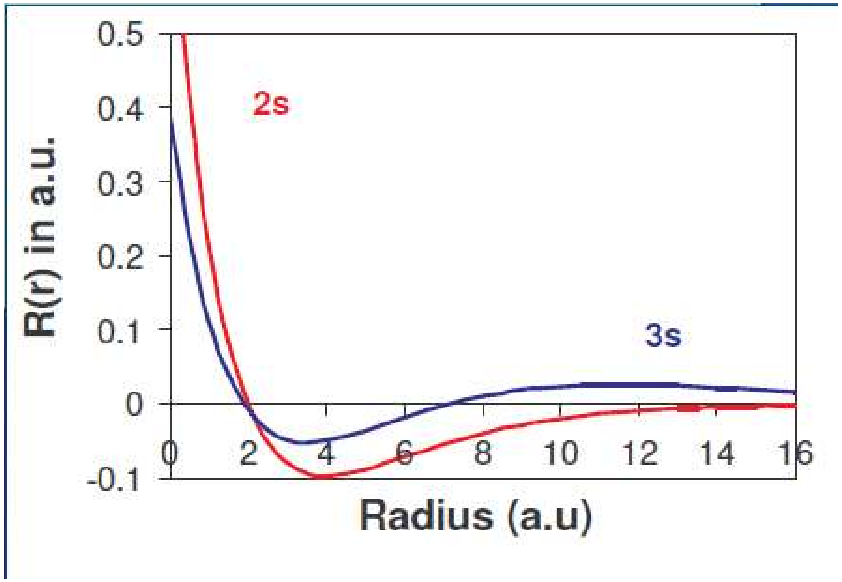
Wave functions of Hydrogen ns orbitals

- For H 2s(r) = $\frac{1}{2\sqrt{2}}(2-r)e^{(-r/2)}$
- For H 3s(r) = $\frac{1}{9\sqrt{3}}\left[6-4r+\left(\frac{2}{3}r\right)^{2}\right]e^{(-r/3)}$
- exponential decreases more

slowly than 1s.

3s changes sign twice with two

- **R(r) = 0 RADIAL NODE**




Piccard
Henriot
Ehrenfest
Herzen
de Donder
Schrödinger
Verschaffelt
Pauli
Heisenberg
Fowler
Brillouin
Debye
Knudsen
Bragg
Kramers
Dirac
Compton
de Broglie
Born
Bohr
Langmuir
Planck
Curie
Lorentz
Einstein
Langevin
Guye
Wilson
Richardson

Fifth Solvay Conference

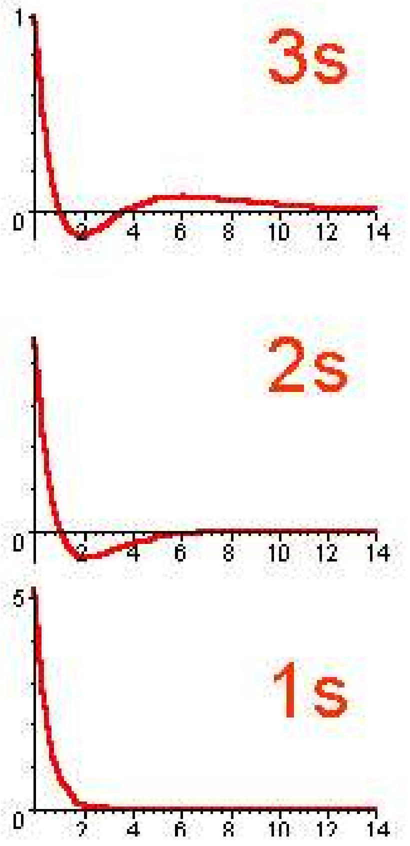
The photo taken of this conference's participants is sometimes entitled "**The Most Intelligent Picture Ever Taken**," for its depiction of the world's leading physicists gathered in one shot.

Perhaps the most famous conference was the fifth Solvay Conference on Physics, which was held from 24 to 29 October 1927. The subject was *Electrons and Photons* and the world's most notable physicists met to discuss the newly formulated quantum theory. The leading figures were Albert Einstein and Niels Bohr. 17 of the 29 attendees were or became Nobel Prize winners, including Marie Curie who, alone among them, had won Nobel Prizes in two separate scientific disciplines. The anti-German prejudice that had prevented Einstein and others from attending the Solvay conferences held after the First World War had melted away. Essentially all of those names who had contributed to the development of the quantum theory were at this Solvay Conference, including Bohr, Born, de Broglie, Dirac, Heisenberg, Pauli, and Schrödinger. Heisenberg commented:

"Through the possibility of exchange between the representatives of different lines of research, this conference has contributed extraordinarily to the clarification of the physical foundations of the quantum theory. It forms, so to speak, the outward completion of the quantum theory."

https://en.wikipedia.org/wiki/Solvay_Conference

Plot of electron wave functions



Electron wave functions

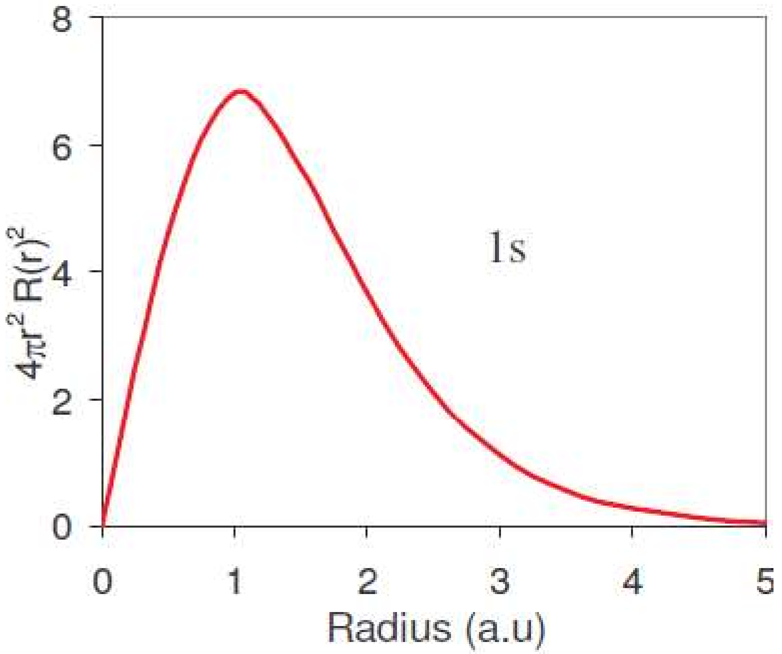
Radial Distribution Function

- Relating the probability of an electron at a point in space to the probability of an electron in a spherical shell at a radius r (an orbit-like picture)
- This is called the Radial Distribution Function (RDF) as in generated by multiplying the probability of an electron at a point which has radius r by the volume of a sphere at a radius of r
- Consider a sphere – volume as we move at a small slice is 4πr2 dr

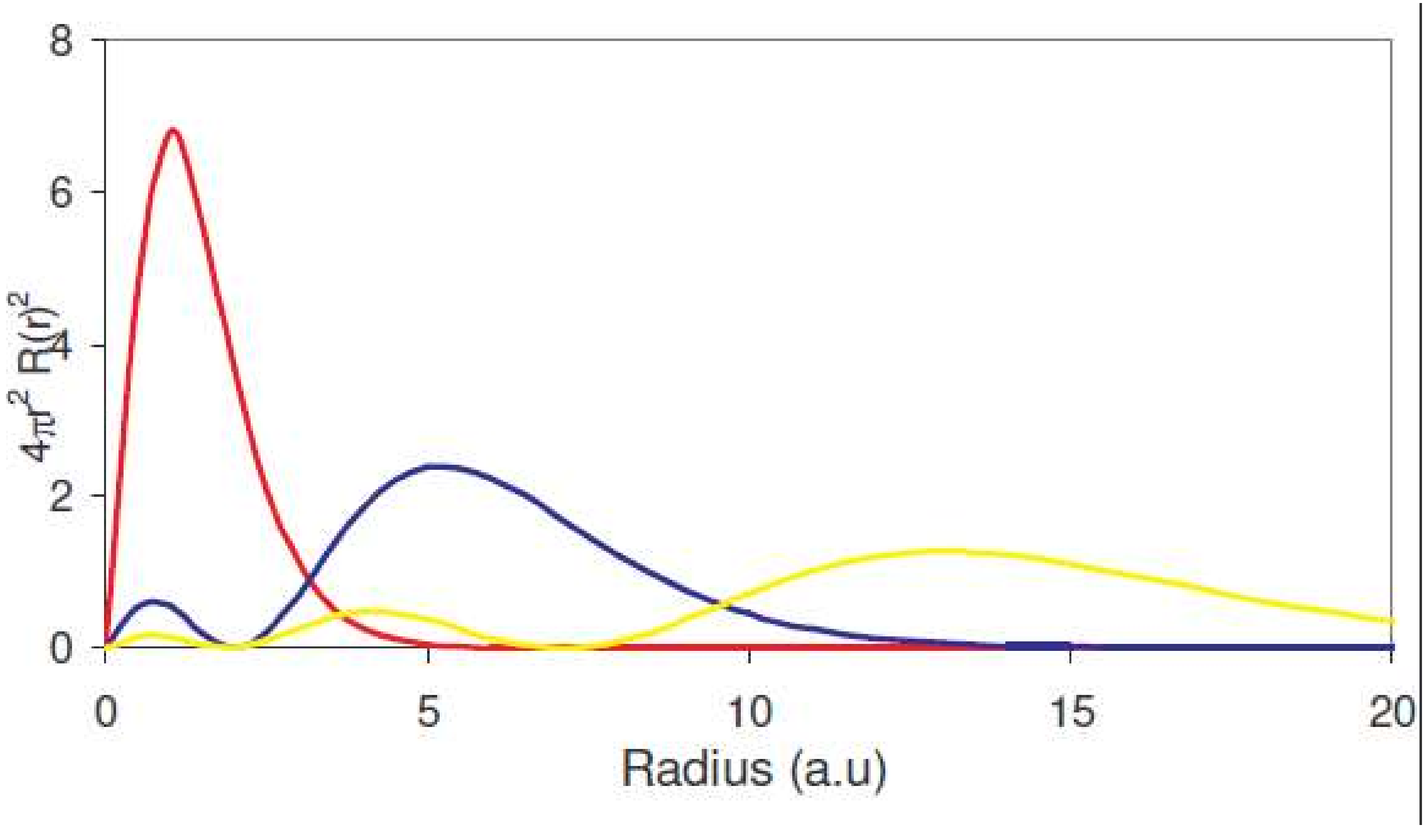
By differentiation,

the volume of a sphere = $4/3\ \pi r^3$

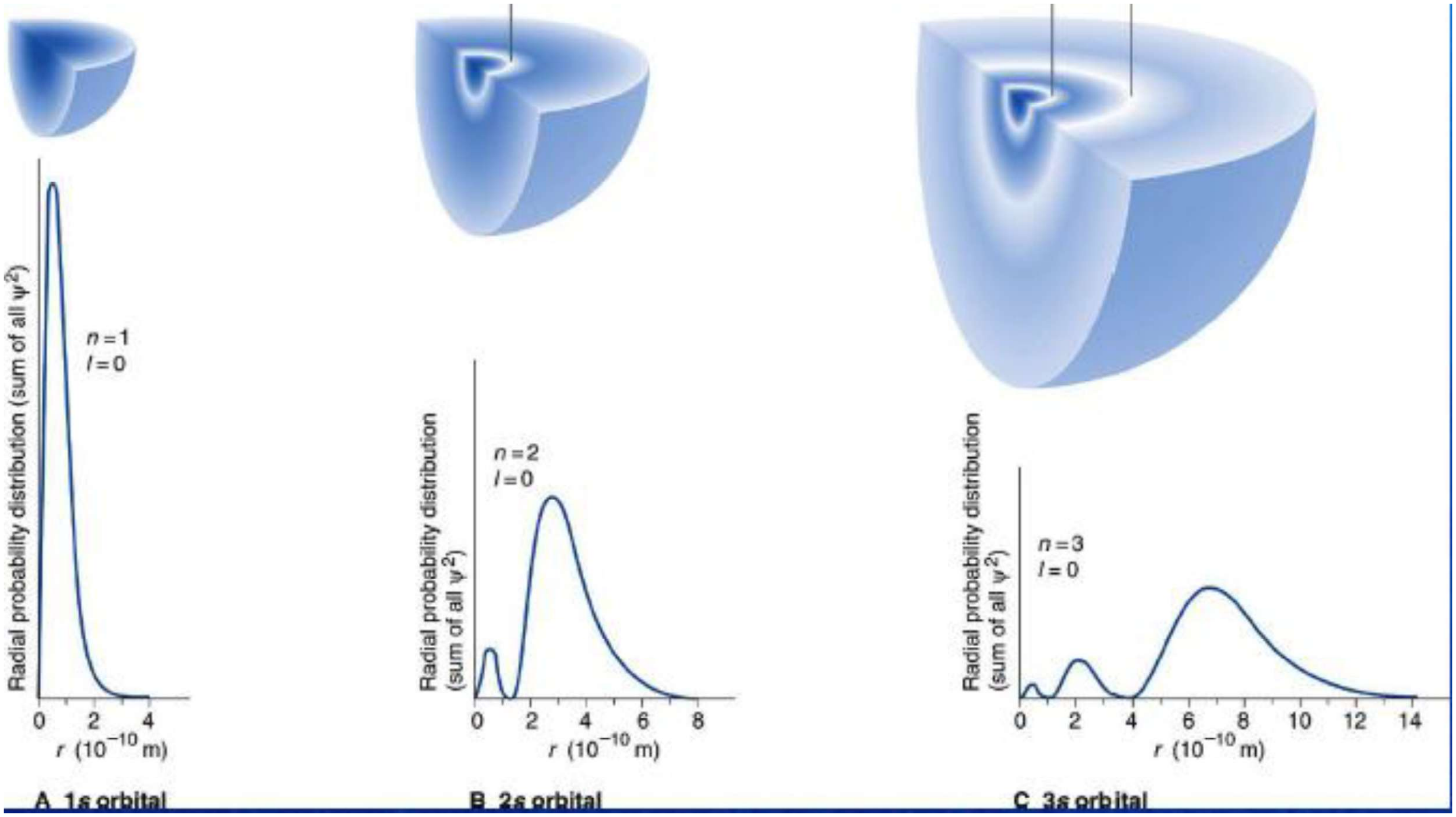
- So RDF(r) = $4\pi r^2\ R(r)^2$
- Maximum for 1s at a_0



RDF



RDF or Probability density plot of 1s, 2s, 3s orbitals



p orbitals

- There are three *p* orbitals for each value of *n* (p_x, p_y, p_z)
- The radial function is the same for all *np* orbitals
- The angular terms are different → different shapes (orientations)
  - Although $2p_x$, $3p_x$, $4p_x$ have the same shape
- Wave function for 2*p* and 3*p* orbitals

$$\rho = \left(\frac{2Z}{na_0}\right)$$

R(r)

$$R(2p) = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a_0}\right)^{3/2} \rho r e^{(-\rho r/2)}$$

$$R(3p) = \frac{1}{9\sqrt{6}}\left(\frac{Z}{a_0}\right)^{3/2} (4 - \rho r)\rho r e^{(-\rho r/2)}$$

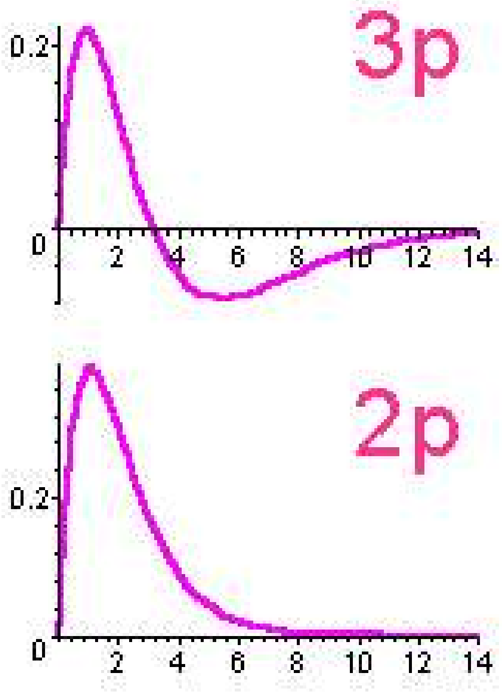
Y(θ ϕ)

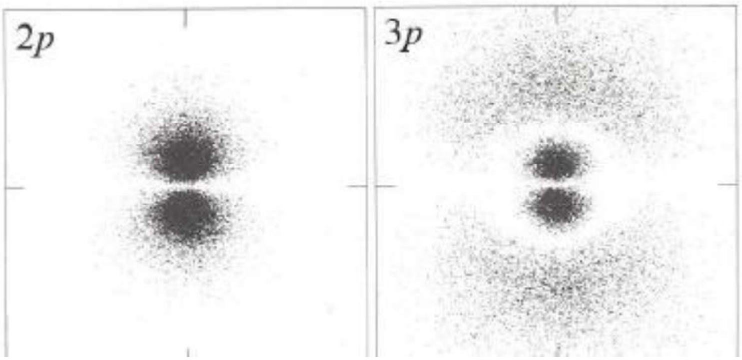
$$Y(p_z) = \left(\frac{3}{4\pi}\right)^{1/2} \cos(\theta)$$

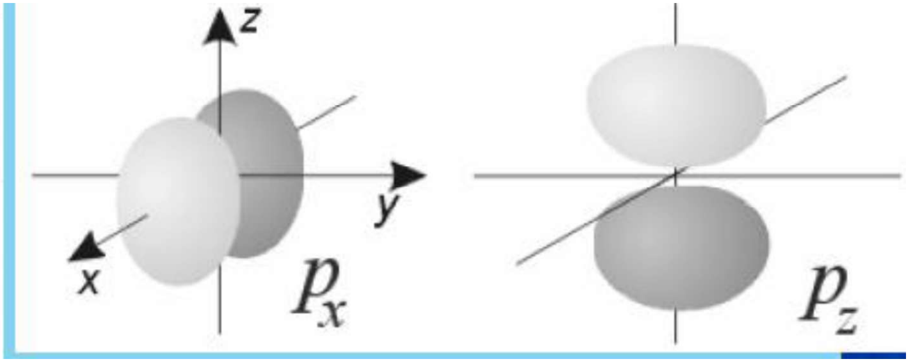
$$Y(p_x) = \left(\frac{3}{4\pi}\right)^{1/2} \sin(\theta)\cos(\phi)$$

$$Y(p_y) = \left(\frac{3}{4\pi}\right)^{1/2} \sin(\theta)\sin(\phi)$$

Plot of radial function of 2p and 3p



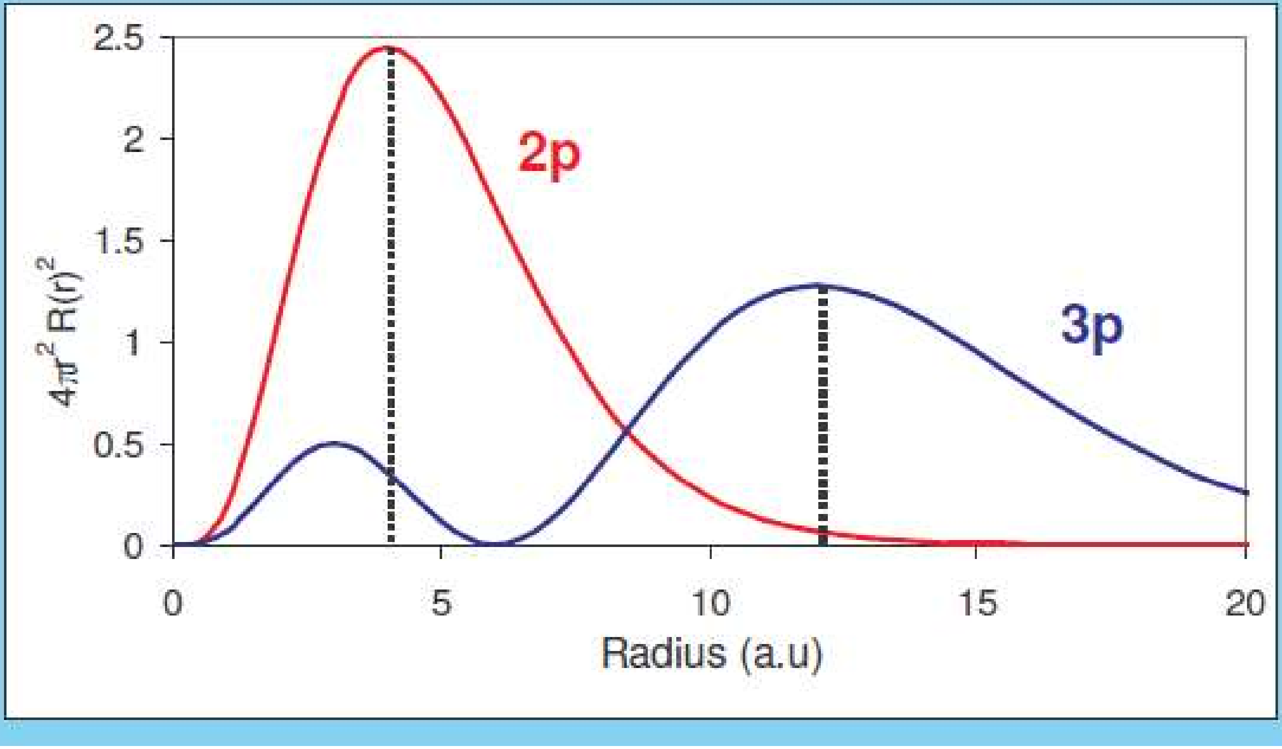

2p
3p


z
y
x
p_x
p_z

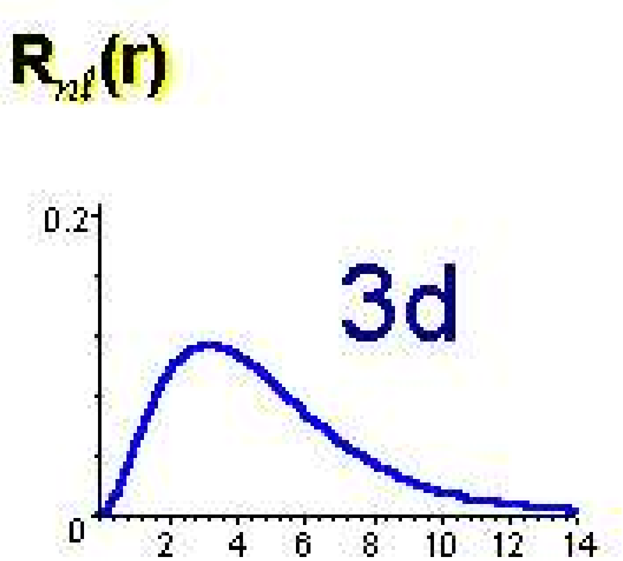
RDF of p orbitals

- Radial distribution function show probability at a given radius

- *2p* function – no nodes, maximum at r = 4 a_0 (same as n = 2 for Bohr model)
- *3p* function – two peaks, maximum at r ≈ 12 a_0



Plot of radial function of 3d

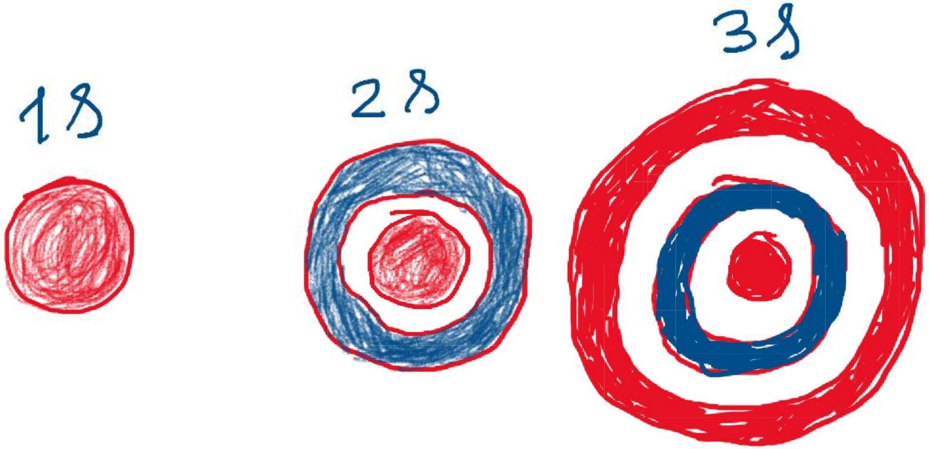


Node

- Nodes are the region with zero probabilities of finding electrons.
- Two types of nodes
- **Radial Node** – obtained from the radial component of the wavefunction
- **Angular Node -** obtained from the angular part of the wavefunction

Radial Nodes

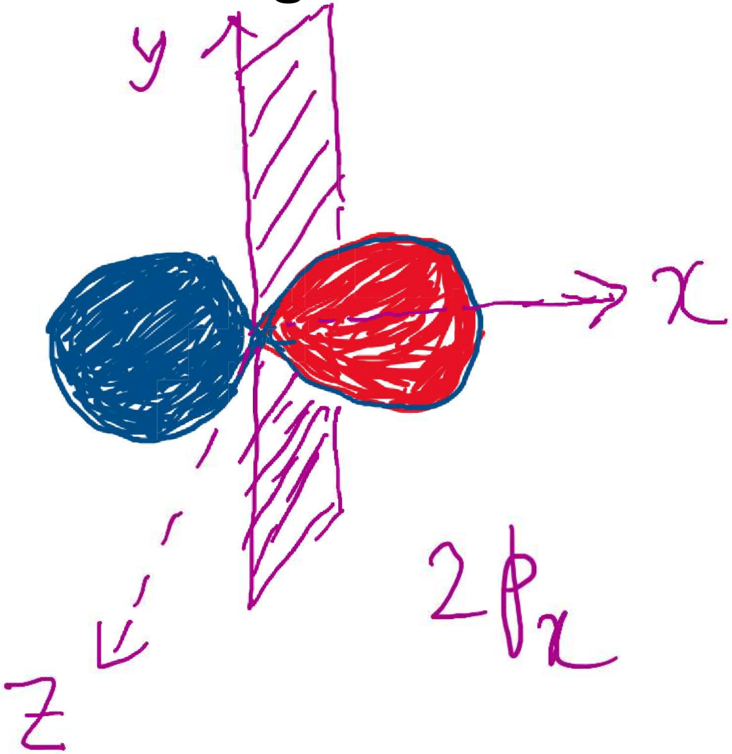
- **The point at which R(r) = 0 (not including the origin) is called a radial node**
- Number of radial node = $\boxed{n - l - 1}$
- **1s = 1 – 0 – 1 = 0**
- **2s = 2 – 0 – 1 = 1**
- **2p = 2 – 1 – 1 = 0**
- **3s = 3 – 0 – 1 = 2**
- **3p = 3 – 1 – 1 = 1**
- **3d = 3 – 2 – 1 = 0**



In general, the more nodes contained within e.g. a set of s orbitals the higher the energy of the orbital – like a wave that crosses the x axis many times

Angular nodes

- Number of angular node= l

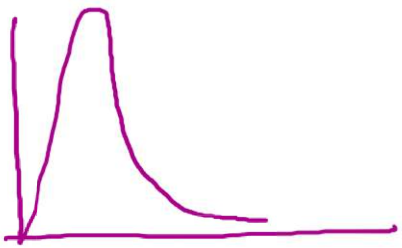


Radial and angular nodes

orbital	*n*	*l*	Radial nodes	Angular nodes	Total nodes
1s	1	0	0	0	0
2s	2	0	1	0	1
3s	3	0	2	0	2
2p	2	1	0	1	1
3p	3	1	1	1	2
4p	4	1	2	1	3
3d	3	2	0	2	2
4d	4	2	1	2	3
5d	5	2	2	2	4
4f	4	3	0	3	3

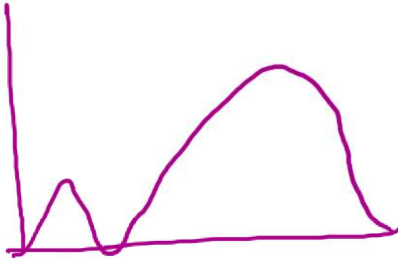
$$1s = \left\{2e^{(-r)}\right\}\left\{\frac{1}{2\sqrt{\pi}}\right\}$$

Radial angular



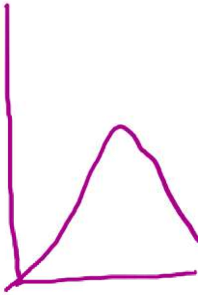
$$2s = \left\{\frac{1}{2\sqrt{2}}(2-r)\,e^{(-r/2)}\right\}\left\{\frac{1}{2\sqrt{\pi}}\right\}$$

Radial angular



$$2p_z = \left\{\frac{1}{2\sqrt{6}}\,r\,e^{(-r/2)}\right\}\left\{\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta\right\}$$

Radial angular



$$3d_{z^2} = \underbrace{\left[\frac{4}{81\sqrt{30}}a_0^{-3/2}\left(\frac{r}{a_0}\right)^2 e^{\left(-r/3a_0\right)}\right]}_{\text{Radial}} \underbrace{\left[\left(\frac{5}{16\pi}\right)^{1/2}\left(3\cos^2\theta - 1\right)\right]}_{\text{Angular}}$$

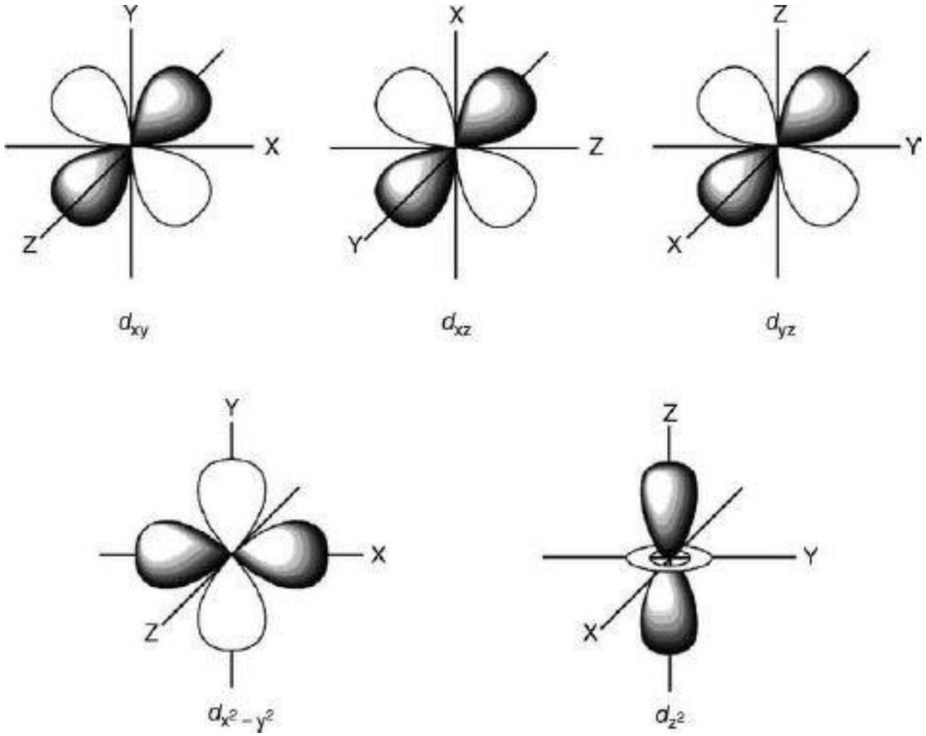
Guess:
No. of radial nodes?
No. of angular nodes?

To find the number of angular nodes, look for the angular part of the wave function. For how many values of θ and ϕ, the angular function becomes zero.

In this case, the angular function can be zero for two values of θ
So, number of angular nodes = 2

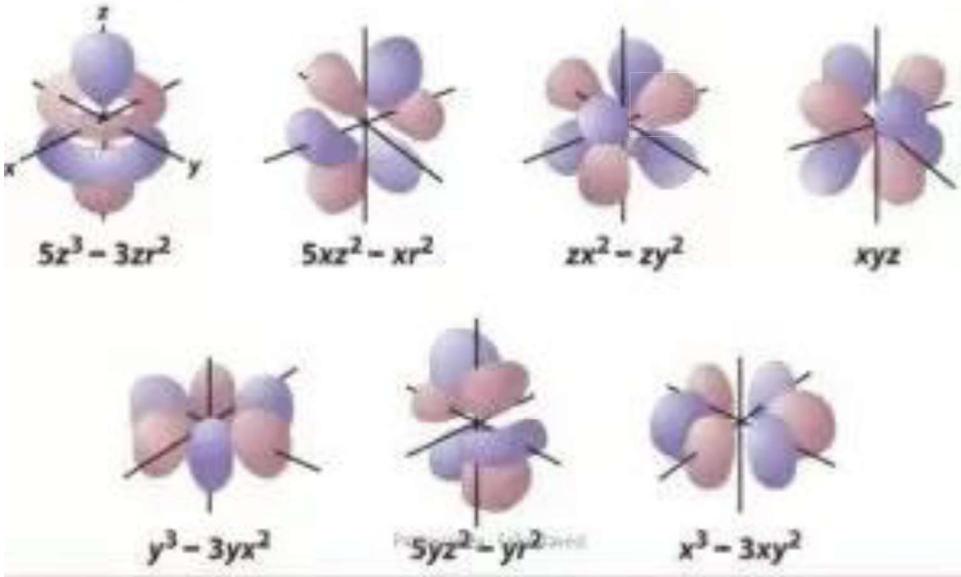
Hints:
To find the number of radial nodes, look for the radial part of the wave function. For how many values of r, the radial function becomes zero.
(r=0 and r= ∞ are not considered)
In this case, the radial function has value zero only at r=0 and r= ∞, and no other values of r can make it zero.
So, number of radial node = 0
(No radial node is present)

Shapes of d orbitals

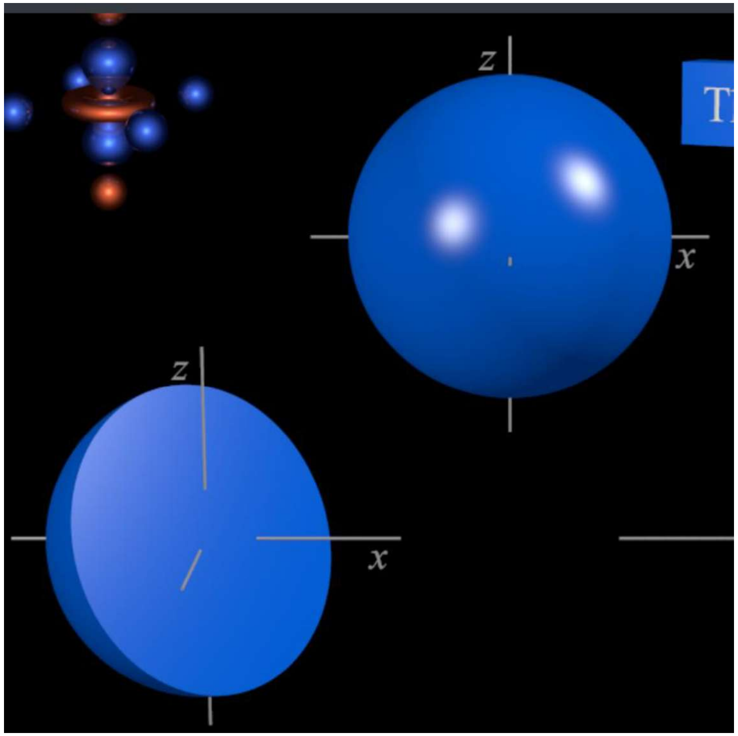


f orbital shapes

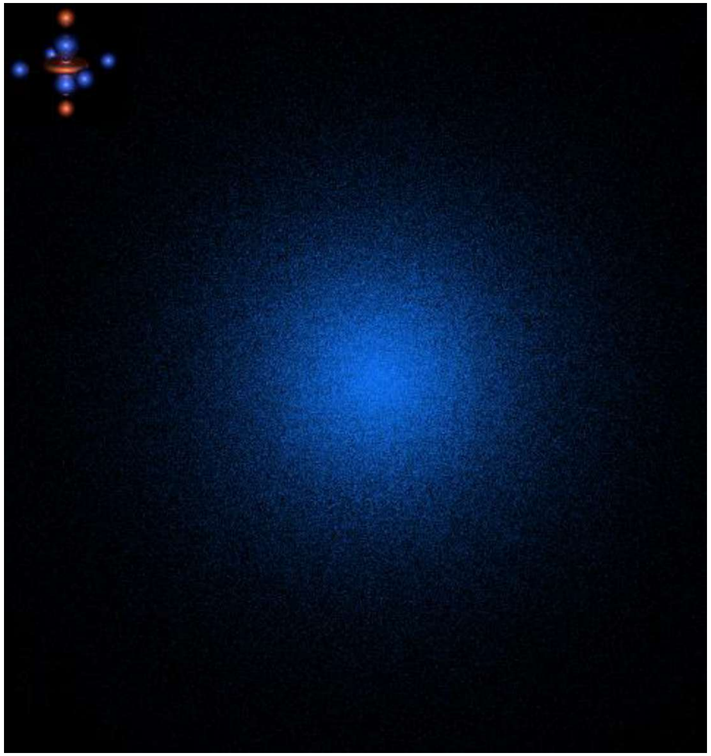
$\lambda = 3$ $m = -3, -2, -1, 0, +1, +2, +3$



1s



No. of radial node??
No. of angular node??

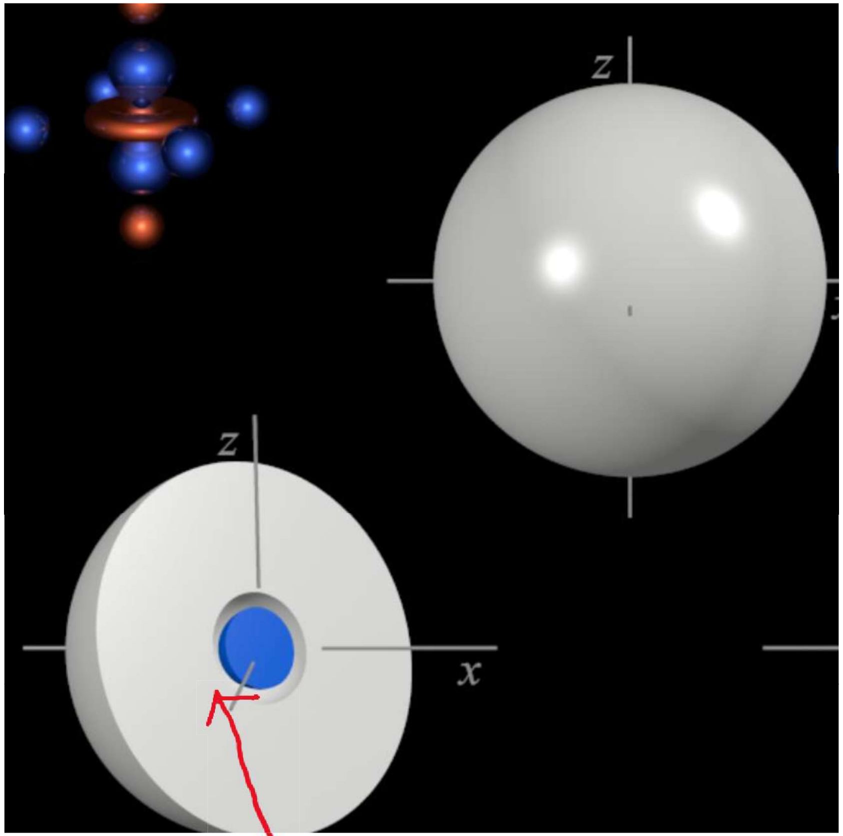


Dot density plot

Reference
Prof. Mark Winter, University of Sheffield
https://winter.group.shef.ac.uk/orbitron/

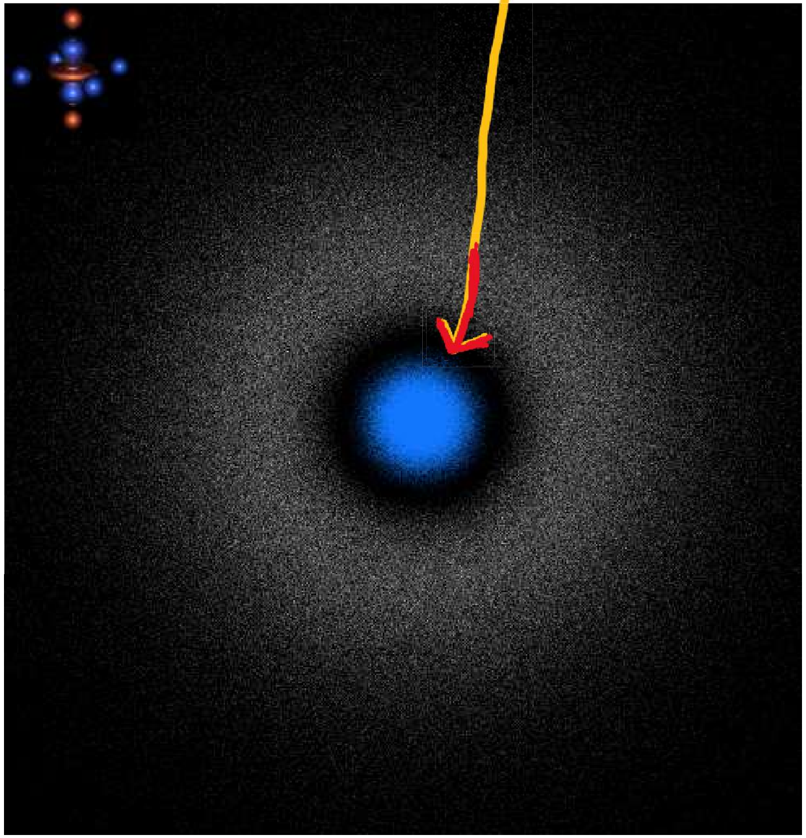
2s

No. of radial node??
No. of angular node??



Radial node

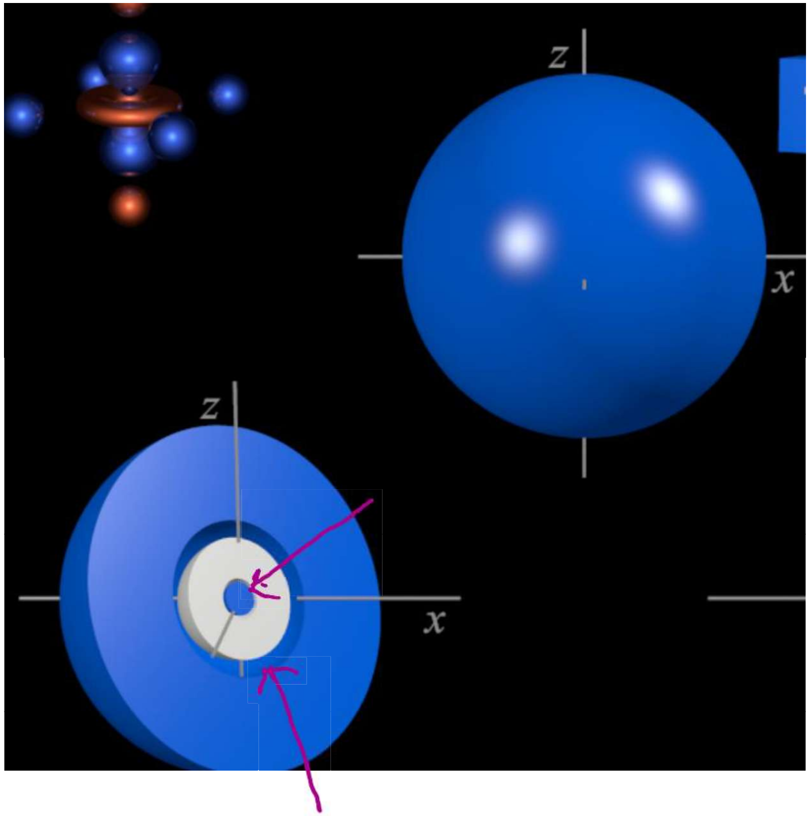
Radial node

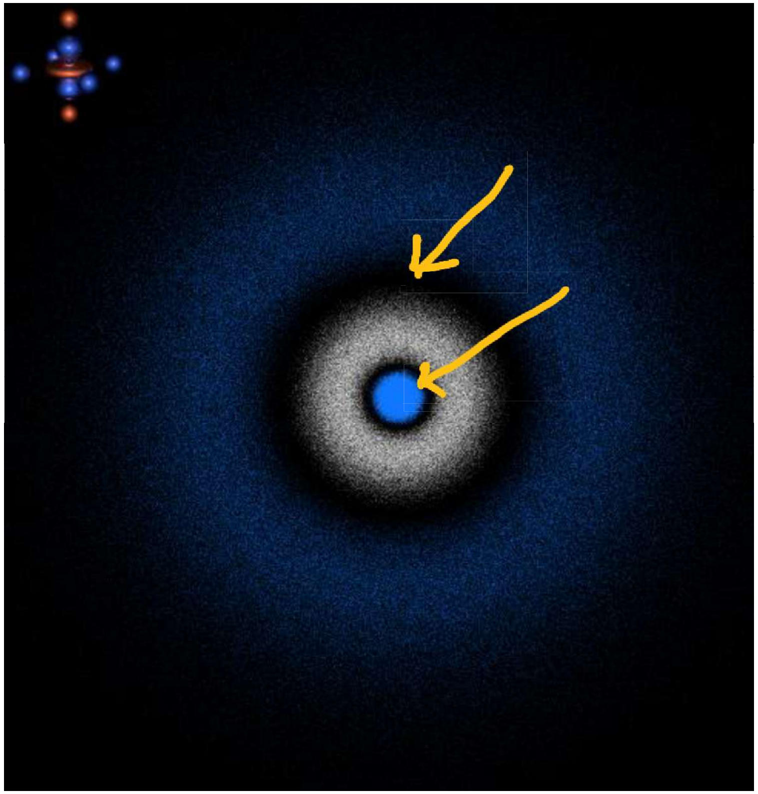


dot-density plot

3s

No. of radial nodes??
No. of angular nodes??





Reference
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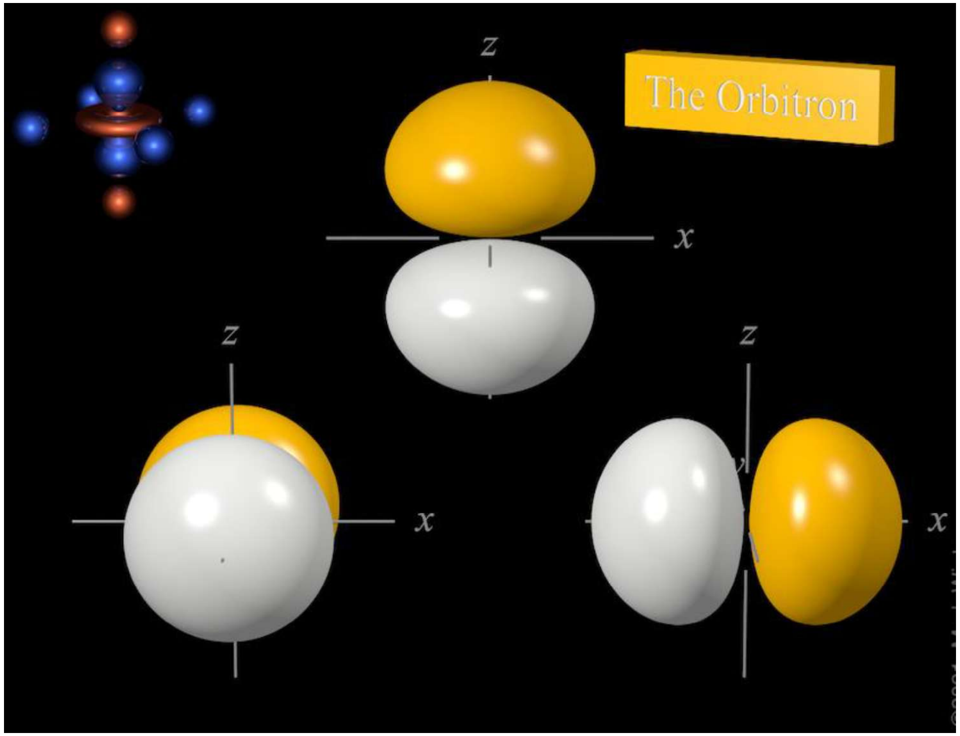
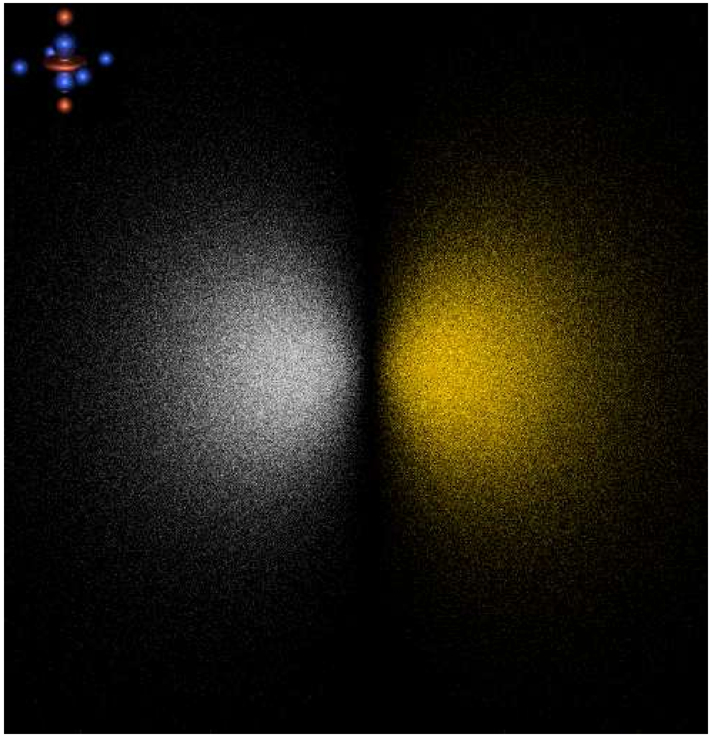
https://winter.group.shef.ac.uk/orbitron/

2p

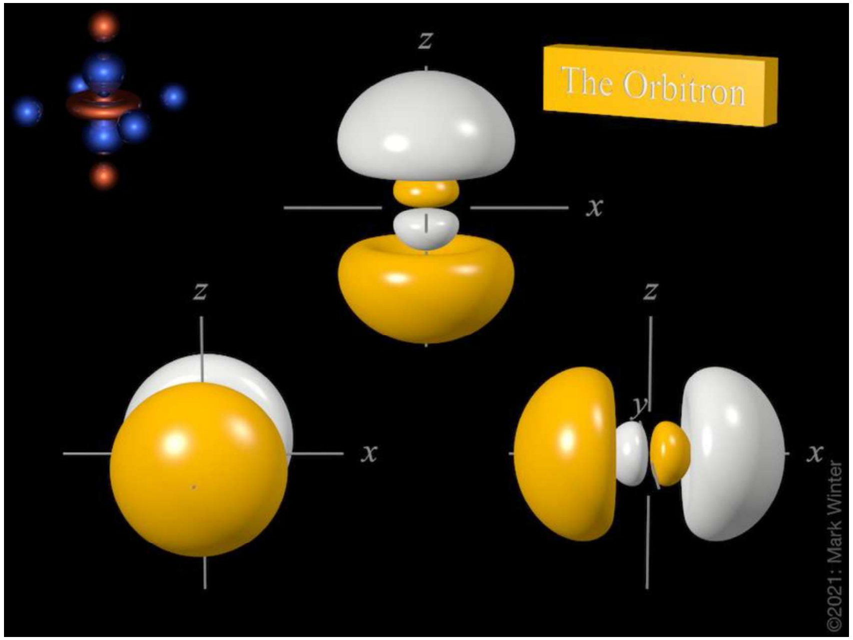
No. of radial node??
No. of angular node??

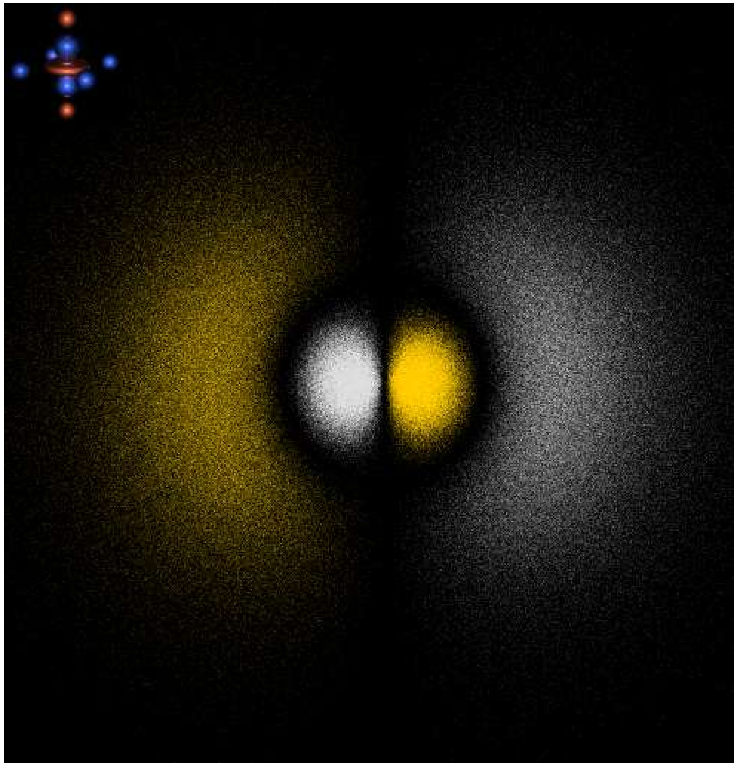




3p

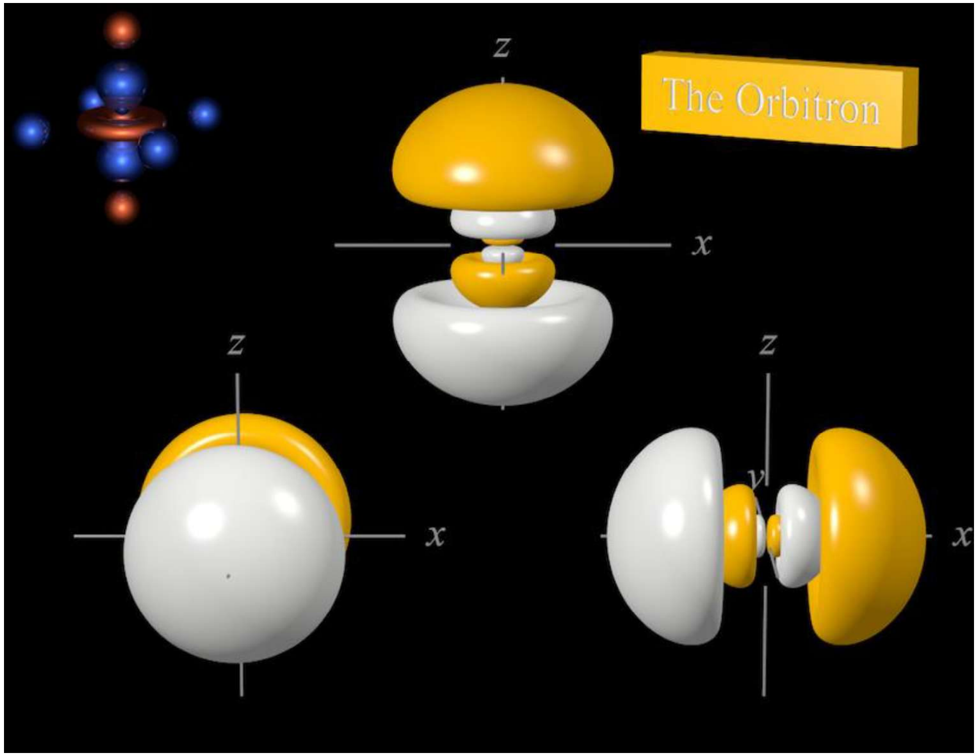
No. of radial node??
No. of angular node??

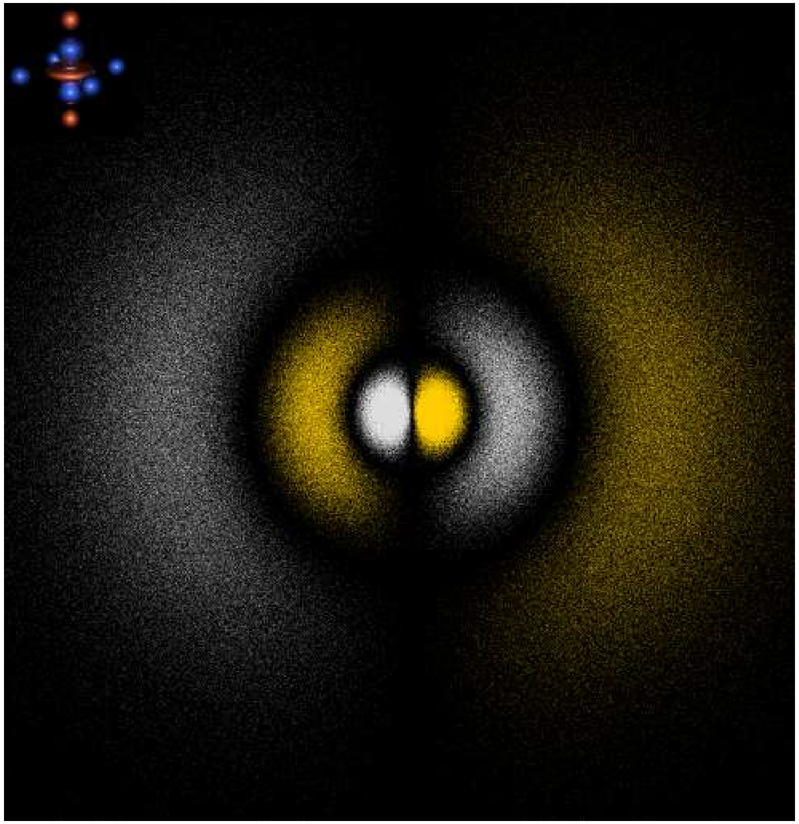




4p

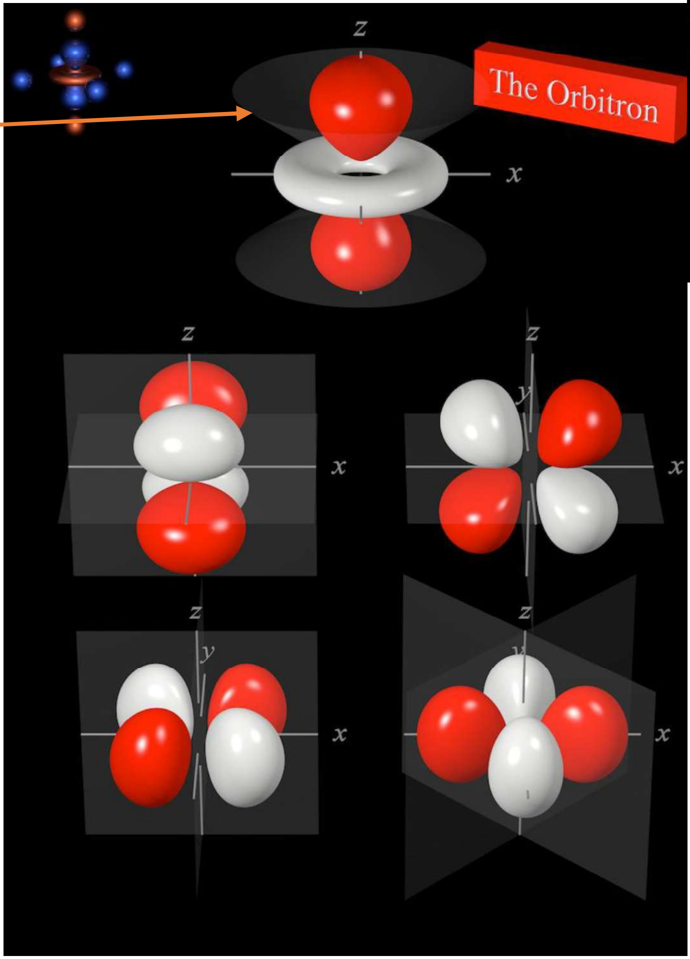
No. of radial node??
No. of angular node??



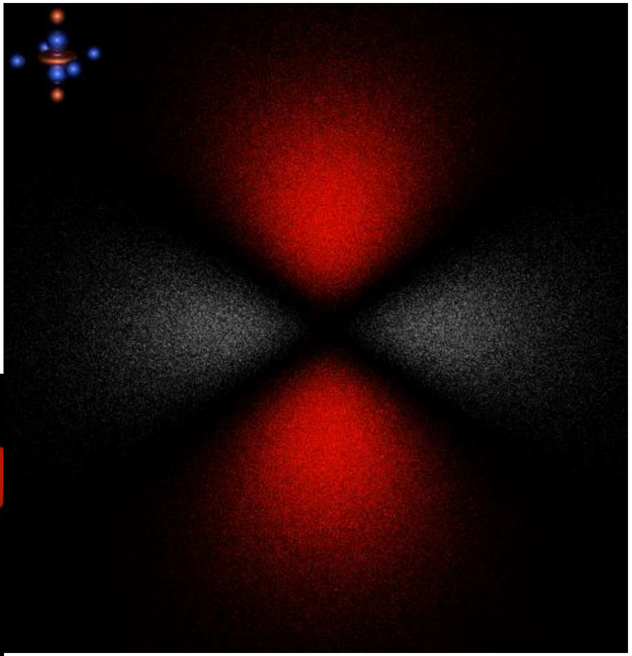


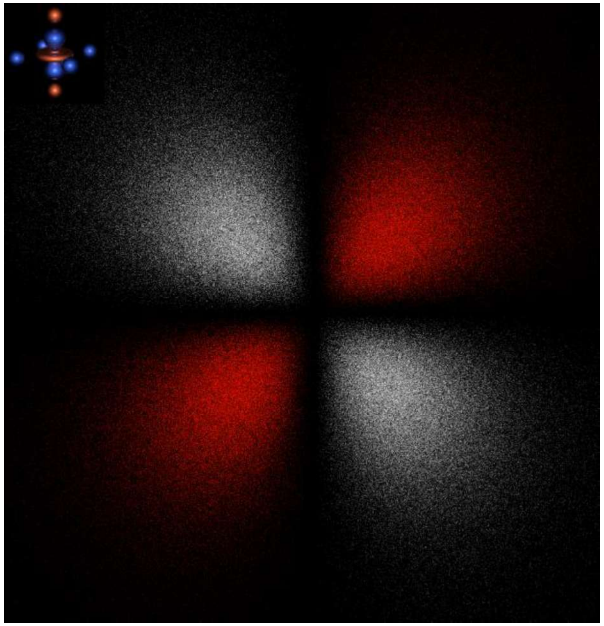
3d

No. of radial node??
No. of angular node??



Cylindrical symmetry





4d

No. of radial node??
No. of angular node??